जून 2009

B.Sc. sem-2nd
paper-VI, unit-1st
complex integration

21 रवि

Note: The above theorem can also be asked in this way.
If $f(z)$ is analytic in a simply connected region D of the complex plane, show that there exists a fn $F'(z)$ analytic in D, and such that $F'(z) = f(z)$ for z in D.

[IAS 70]

Theorem 5 **Taylor's theorem** :→

If a function $f(z)$ is analytic within a circle C with its centre $z = a$ and radius R, then at every point z inside C

22 सोम

$$f(z) = \sum_{n=0}^{\infty} f^n(a) \frac{(z-a)^n}{\lfloor n}$$

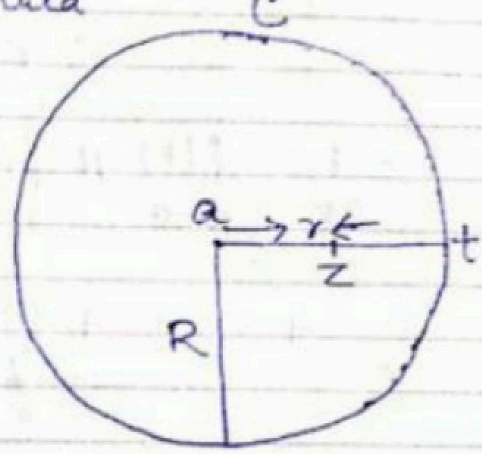
Or

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \text{ where } a_n = \frac{f^n(a)}{\lfloor n}$$

Let $f(z)$ be analytic within a circle C whose equation is $|t-a| = R$.

Let Z be any point within C such that $|z-a| = r < R$

By Cauchy's Integral formula



$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{(t-z)}\,dt$$

$$= \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{t-a-z+a}$$

$$= \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{(t-a)-(z-a)}$$

$$= \frac{1}{2\pi i}\int_C \frac{f(t)}{(t-a)\left\{1-\frac{z-a}{t-a}\right\}}\,dt$$

$$= \frac{1}{2\pi i}\int_C \frac{f(t)}{t-a}\left\{1-\frac{z-a}{t-a}\right\}^{-1} dt \quad \text{—— (1.)}$$

Applying the formula

$$\frac{1}{1-b} = (1-b)^{-1} = 1+b+b^2+b^3+ - +b^{n-1}+b^n+b^{n+1}+b^{n+2} + - - - - - - .$$

$$= 1+b+b^2+b^3+ - - - + b^{n+1}(1+b+b^2+b^3+ - - \infty)$$

$$= 1+b+b^2+b^3+ - - - + b^{n+1}(1-b)^{-1}$$

रवि	सोम	मंगल	बुध	गुरु	शुक्र	शनि
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30	31	

जुलाई 2009

टिप्पणी

So we have from (1.)

25 गुरु

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{(t-a)}\left[1+\frac{z-a}{t-a}+\left(\frac{z-a}{t-a}\right)^2+\cdots + \left(\frac{z-a}{t-a}\right)^n+\left(\frac{z-a}{t-a}\right)^{n+1}\left(1-\frac{z-a}{t-a}\right)^{-1}\right]dt$$

$$= \frac{1}{2\pi i}\int_C \frac{f(t)}{t-a}\,dt + (z-a)\frac{1}{2\pi i}\int \frac{f(t)}{(t-a)^2}\,dt + (z-a)^2\frac{1}{2\pi i}\int_C \frac{f(t)}{(t-a)^3}\,dt$$

$$+\cdots + \frac{(z-a)^{n+1}}{2\pi i}\int_C \frac{f(t)\,dt}{(t-a)^{n+1}\left(1-\frac{z-a}{t-a}\right)}$$

$$= \frac{1}{2\pi i}\int_C \frac{f(t)}{t-a}\,dt + \frac{z-a}{2\pi i}\int_C \frac{f(t)}{(t-a)^2}\,dt + \frac{(z-a)^2}{2\pi i}\int_C \frac{f(t)}{(t-a)^3}\,dt$$

26 शुक्र

$$+\cdots + \frac{(z-a)^n}{2\pi i}\int_C \frac{f(t)\,dt}{(t-a)^{n+1}}$$

$$+ \frac{(z-a)^{n+1}}{2\pi i}\int_C \frac{f(t)}{(t-a)^{n+2}\left(\frac{t-\cancel{a}-z+\cancel{a}}{t-a}\right)}$$

Or

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-a}\,dt + \frac{z-a}{2\pi i}\int \frac{f(t)}{(t-a)^2}\,dt + \frac{(z-a)^2}{2\pi i}\int_C \frac{f(t)\,dt}{(t-a)^3}$$

$$+\cdots + \frac{(z-a)^{n+1}}{2\pi i}\int_C \frac{f(t)}{(t-a)^{n+2-1}(t-z)} \qquad (2.)$$

टिप्पणी

मई 2009

रवि	सोम	मंगल	बुध	गुरु	शुक्र	शनि
31					1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28	29	30

We know

$$f^n(a) = \frac{\lfloor n}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}}$$

$$\therefore \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}} = \frac{f^n(a)}{\lfloor n} \quad \text{---} \quad (3)$$

From (2)

$$f(z) = f(a) + \frac{z-a}{\lfloor 1} f'(a) + \frac{(z-a)^2}{\lfloor 2} f''(a) + \cdots$$

$$+ \cdots + \frac{(z-a)^n}{\lfloor n} f^n(a) + U_{n+1} \quad \text{-----} \quad (4)$$

where $U_{n+1} = \dfrac{(z-a)^{n+1}}{2\pi i}\displaystyle\int_C \frac{f(t)\,dt}{(t-z)(t-a)^{n+1}}$

$$|U_{n+1}| \le \frac{|z-a|^{n+1}}{2\pi}\int_C \frac{|f(t)|\,|dt|}{(|t|-|z|)\,|t-a|^{n+1}}$$

$$\le \frac{M}{2\pi}\left(\frac{r}{R}\right)^{n+1}\frac{1}{R-r}\,2\pi R$$

where $M = \max|f(t)|$ on C

or $|U_{n+1}| \le M\left(\dfrac{r}{R}\right)^{n+1}\dfrac{1}{1-\frac{r}{R}} \to 0$ as $n \to \infty$

$$\left[\because \lim_{n\to\infty}\left(\frac{r}{R}\right)^{n+1} = 0 \text{ as } \frac{r}{R} < 1\right]$$

29 सोम $\therefore \underset{n\to\infty}{Lt}\ U_{n+1} = 0$ making $n \to \infty$ in (4).

we get—

$$f(z) = \underset{n\to\infty}{Lt}\left[f(a) + \frac{z-a}{\lfloor 1}\, f'(a) + \frac{(z-a)^2}{\lfloor 2}\, f''(a) + \cdots + \frac{(z-a)^n}{\lfloor n}\, f^n(a) \right]$$

$$= \sum_{n=0}^{\infty} \frac{(z-a)^n}{\lfloor n} f^n(a) = \sum_{n=0}^{\infty} (z-a)^n \frac{f^n(a)}{\lfloor n}$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n \text{ —————— } (5)$$

30 मंगल where $a_n = \dfrac{f^n(a)}{\lfloor n}$

found

Deduction: Since z is a pt within the circle $|t-a| = R$ such that $|z-a| = r < R$ so that we can take $z = a+h$ or $h = z-a$

from (5)

$$f(z) = \sum_{n=0}^{\infty} \frac{h^n}{\lfloor n} f^n(a)$$

(6)

$$= f(a) + h\, f'(a) + \frac{h^2}{\lfloor 2} f''(a) + \cdots$$

This is alternative form of Taylor's Series.

(ii) If we write $a = 0$ in (5)

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$= \sum_{n=0}^{\infty} \frac{f^n(a)}{\lfloor n} z^n$$

This is known as Maclaurin's Series.